Probability theory

(1) Probability spaces

Study questions

1. Write down the definition of a probability space.

2. Give two interpretations for the probability $\mathbb{P}(A)$ of an event A.

3. Write down the definition of the independence of two events A and B and the definition of the independence of a set of events $\{A_i | i \in I\}$ with index set I.

4. Write down the definition of the conditional probability of an event A given an event B.

5. What is the conditional probability of an event A given an event B, if A and B are independent events? Justify your answer.

6. Write down and prove the law of total probability.

7. Write down and prove Bayes theorem.

Exercises (Theory)

1. Develop a probability space model of throwing two dice (DeGroot and Schervish, 2012, Example 1.6.5).

2. Develop a probability space model of tossing a fair coin twice. Consider the events “heads appears on the first toss”, “heads appears on the second toss”, and “both tosses have the same outcome”. Show that these three events are pairwise independent, but that all three events are not independent (DeGroot and Schervish, 2012, Example 2.2.4).

Exercises (Programming)

1. Consider the probability space model of tossing a fair dice. Let $A = \{2, 4, 6\}$ and $B = \{1, 2, 3, 4\}$ be two events. Then, $\mathbb{P}(A) = 1/2$, $\mathbb{P}(B) = 2/3$ and $\mathbb{P}(A \cap B) = 1/3$. Since $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$, the events A and B are independent. Simulate draws from the outcome space and verify that $\hat{\mathbb{P}}(A \cap B) = \hat{\mathbb{P}}(A)\hat{\mathbb{P}}(B)$, where $\hat{\mathbb{P}}(E)$ denotes the proportion of times an event E occurs in the simulation. Next, identify two events A and B that are not independent. Analytically, evaluate $\mathbb{P}(A)$, $\mathbb{P}(B)$, $\mathbb{P}(A \cap B)$, $\mathbb{P}(A|B)$ and $\mathbb{P}(B|A)$ and verify these values using the simulation. Document your results.

2. Consider the probability space model of tossing a fair coin twice and the events “heads appears on the first toss”, “heads appears on the second toss”, and “both tosses have the same outcome”. Simulate draws from the outcome space and verify that (1) the probability of each of the events is 1/2, (2) the probability for the co-occurrence of pairs of the events, as well as of all events is 1/4.