(4) Random variable transformations

Study questions

1. Let X be a discrete random variable and $Y = f(X)$ a transformation of X.Express the PMF p_Y of Y in terms of the distribution of X.

2. Let X be a continuous random variable with PDF p_X and let $Y = f(X)$ be a transformation of X. Write down direct calculation procedure to derive the PDF p_Y of Y.

3. Write down the probability integral transform theorem.

4. How can a uniform random number generator be used to create random numbers with arbitrary distribution?

5. Write down the univariate probability density function transform theorem.

6. Write down the univariate probability density function transform theorem for linear functions.

7. Write down the Z-transformation of univariate Gaussian random variable.

8. Write down the multivariate probability density function transform theorem.

9. Write down the multivariate probability density function transform theorem for linear functions.

10. Let an n-dimensional random vector X be distributed according to multivariate Gaussian distribution, $X \sim N(\mu, \Sigma)$. Let $A \in \mathbb{R}^{n\times n}$ be a matrix of full column-rank. How is the random vector $Y := AX$ distributed?

Exercises (Theory)

1. Let $X_1, ..., X_n \sim \text{Bern}(\mu)$. Show that $Y = \sum_{i=1}^{n} X_i$ is distributed according to a Binomial distribution with parameters n and μ, which we denote by $Y \sim \text{Bin}(n, \mu)$. Clarify, why there are $\binom{n}{k}$ distinct possible values for a binary vector of n entries to comprise k ones and $n - k$ zeros, where $\binom{n}{k}$ denotes the Binomial coefficient (DeGroot and Schervish, 2012, Theorem 3.9.2).

2. Assume that $X_1, ..., X_n$ form a random sample of size n from a distribution with PDF p and CDF P. Let $Y_{\max} := \max\{X_1, ..., X_n\}$ and $Y_{\min} = \min\{X_1, ..., X_n\}$ denote the maximum and minimum of $X_1, ..., X_n$, respectively. Evaluate the distributions of $Y_{\max}$ and $Y_{\min}$ in general, and for the case that $X \sim N(\mu, \sigma^2)$ (DeGroot and Schervish, 2012, Example 3.9.6).

3. Let two random variables X_1 and X_2 have a joint PDF p_{X_1,X_2} and let $Y := a_1X_1 + a_2X_2 + b$ with $a_1 \neq 0$. Show that the Y has a continuous distribution with PDF

$$p_Y(y) := \int_{-\infty}^{\infty} p_{X_1,X_2}\left(\frac{y - b - a_2x_2}{a_1}, x_2\right)\frac{1}{|a_1|}\,dx_2 \tag{1}$$

(DeGroot and Schervish, 2012, Theorem 3.9.4). In addition, write down p_Y for X_1, X_2 independent, $a_1 := a_2 := 1$ and $b := 0$ DeGroot and Schervish (c.f. 2012, Definition 3.9.1)

Exercises (Programming)

1. Write a program that generates pseudo-random numbers from an exponential distribution using a uniform pseudo-random number generator and the probability integral transform theorem. Visualize and document your results.

2. Let $X \sim N(0, 1)$ and let $Y = \exp(X)$. Evaluate the PDF of Y analytically and verify your evaluation using a simulation based on drawing random numbers from $N(0, 1)$. Visualize and document your results.

3. Let $X \sim N(0, 1)$ and let $Y = X^2$. By simulation, validate that Y is distributed according to a chi-squared distribution with one degree of freedom. Next, let $X_1, ..., X_10 \sim N(0, 1)$ and let $Y = \sum_{i=1}^{10} X_i^2$. By simulation, validate that Y is distributed according to a chi-squared distribution with ten degrees of freedom.