(5) Expectation, (co)variance, inequalities, limits

Study questions

1. Discuss the intuition of the expected value of a random variable.
2. What does it mean for the expectation of a random variable to exist?
3. Compute the expectation of a Bernoulli random variable.
4. State the linearity and multiplication properties of expectations.
5. Write down $(E)(X^2)$ in terms of the variance and expectation of the random variable X.
6. For constant a, what is $\mathbb{V}(aX)$?
7. Write down the definition of the covariance and correlation of two random variables X and Y.
8. Express the covariance of two random variables X and Y in terms of expectations.
9. What is the variance of the sum of two random variables X and Y, if X and Y are independent and in general?
10. What is the variance of the difference of two random variables X and Y, if X and Y are independent and in general?
11. Write down the Markov inequality.
12. Write down the Chebychev inequality.
13. Write down Jensen's inequality for concave functions.
14. Write down the Cauchy-Schwarz inequality.
15. Write down the Correlation inequality.
16. Write down the definition of convergence in probability.
17. Write down the definition of almost-sure convergence.
18. Write down the definition of convergence in distribution.
19. Write down the Weak Law of Large Numbers.
20. Write down the Strong Law of Large Numbers.
21. Write down the Lindenberg-Lévy form of the Central Limit Theorem.
22. Write down the Liapunov form of the Central Limit Theorem.

Exercises (Theory)

1. For two continuous random variables X_1, X_2 and two constants a_1, b_2, show that

$$\mathbb{E}(a_1 X_1 + a_2 X_2) = a_1 \mathbb{E}(X_1) + a_2 \mathbb{E}(X_2) \tag{2}$$

(cf. DeGroot and Schervish (2012, pp. 217 - 219)). In addition, show that for two independent continuous random variables X_1, X_2 it holds that

$$\mathbb{E}(X_1 X_2) = \mathbb{E}(X_1)\mathbb{E}(X_2) \tag{3}$$

(cf. DeGroot and Schervish (2012, Theorem 4.2.6)).

2. Show that for a random variable X and as constant a it holds that

$$\mathbb{V}(aX) = a^2\mathbb{V}(X) \tag{4}$$

(DeGroot and Schervish (2012, Theorem 4.3.4)) and that for and that for independent random variables X_1, X_2 and constants a_1, a_2 it holds that

$$\mathbb{V}(a_1X_1 + a_2X_2) = a_1^2\mathbb{V}(X_1) + a_2^2\mathbb{V}(X_2). \tag{5}$$

(cf. DeGroot and Schervish (2012, Theorem 4.3.5, Corollary 4.3.1)).

3. Show that for two random variables X and Y it holds that

$$\mathbb{V}(X + Y) = \mathbb{V}(X) + \mathbb{V}(Y) + 2\mathbb{C}(X, Y) \tag{6}$$

and that

$$\mathbb{V}(X - Y) = \mathbb{V}(X) + \mathbb{V}(Y) - 2\mathbb{C}(X, Y) \tag{7}$$

(cf. DeGroot and Schervish (2012, Theorem 4.6.6, Corollary 4.6.1)). Also show that for n random variables $X_1, ..., X_n$ it holds that

$$\mathbb{V}\left(\sum_{i=1}^{n} a_iX_i\right) = \sum_{i=1}^{n} a_i^2\mathbb{V}(X_i) + 2\sum_{j=1}^{n}\sum_{i\leq j}\mathbb{C}(X_i, X_j) \tag{8}$$

(cf. DeGroot and Schervish (2012, Theorem 4.6.7)).

Exercises (Programming)

1. Write a simulation that validates the Weak Law of Large Numbers. Visualize and document your results.
2. Write a simulation that validates the Strong Law of Large Numbers. Visualize and document your results.
3. Write a simulation that validates the Central Limit Theorem in the Lindenberg-Lévy form. Visualize and document your results.
4. Write a simulation that validates the Central Limit Theorem in the Liapunov form. Visualize and document your results.