(7) Finite-sample estimator properties

Study questions

1. Define the bias of an estimator. When is an estimator unbiased?
2. Write down the definition of the variance of an estimator.
3. Define the standard error of an estimator.
4. Define the standard error of the mean.
5. Write down the definitions of the score function of a random variable/sample.
6. Write down the definitions of the Fisher information function of a random variable/sample.
7. Write down the definitions of the expected Fisher information of a random variable/sample.
8. What are the expected value and the variance of the score function of a random variable?
9. Formulate the Cramér-Rao bound theorem.
10. Write down the bias-variance decomposition for the mean squared error of an estimator.

Exercises (Theory)

1. Evaluate the Fisher information and the expected Fisher information for for the expectation and variance parameters of a univariate Gaussian based on a random sample $X_1, ..., X_n \sim N(\mu, \sigma^2)$ (Held and Sabanés Bové, 2014, Examples 2.9 and 4.3).
2. Let X be a Poisson-distributed random variable with parameter $e\lambda$ for a known offset $e > 0$ and an unknown parameter λ. The maximum likelihood estimator of λ is given by $\hat{\lambda} = X/e$. Show that $\hat{\lambda}$ attains the Cramér-Rao bound (Held and Sabanés Bové, 2014, Example 4.11)
3. Let $X_1, ..., X_n \sim N(\mu, \sigma^2)$. Show that the estimator

$$\hat{\sigma}^2 := \frac{1}{n+1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2 \tag{1}$$

   for the variance parameter σ^2 has a smaller mean squared error than both the maximum likelihood estimator of σ^2 and the sample variance S^2 (DeGroot and Schervish, 2012, Example 8.7.6).

Exercises (Programming)

1. For $X_1, ..., X_n \sim N(\mu, \sigma^2)$ implement a simulation which validates the unbiasedness of the sample mean, the unbiasedness of the sample variance, the biasedness of the sample standard deviation, and the biasedness of the maximum likelihood variance parameter estimator.
2. For $X_1, ..., X_n \sim B(\mu)$ implement a simulation which validates the unbiasedness of the sample mean, the unbiasedness of the sample variance, the biasedness of the sample standard deviation, and the biasedness of the maximum likelihood variance parameter estimator.
3. Let $X_1, ..., X_n \sim N(\mu, \sigma^2)$ implement a simulation that validates the bias-variance decompositions of the mean squared errors of the maximum likelihood estimator of σ^2, the sample variance S^2, and the estimator $\hat{\sigma}^2$ introduced in the first theoretical exercise above (DeGroot and Schervish, 2012, Example 8.7.6).