(1) Introduction

1. Sample a univariate Gaussian using scipy.stats.

2. Evaluate the PDF of a univariate Gaussian using scipy.stats.

3. Visualize the PDF of a univariate and a normalized sample histogram of samples from a univariate Gaussian with identical parameters on top of each other using Matplotlib.

(2) Probability spaces

1. (Dice experiment 1) Consider the probability space model of tossing a fair dice. Let $A = \{2, 4, 6\}$ and $B = \{1, 2, 3, 4\}$ be two events. Then, $\mathbb{P}(A) = 1/2$, $\mathbb{P}(B) = 2/3$ and $\mathbb{P}(A \cap B) = 1/3$. Since $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$, the events A and B are independent. Simulate draws from the outcome space and verify that $\hat{\mathbb{P}}(A \cap B) = \hat{\mathbb{P}}(A)\hat{\mathbb{P}}(B)$, where $\hat{\mathbb{P}}(E)$ denotes the proportion of times an event E occurs in the simulation.

2. (Dice experiment 2) Consider the probability space model of tossing a fair dice. Identify two events A and B that are not independent. Analytically, evaluate $\mathbb{P}(A)$, $\mathbb{P}(B)$, $\mathbb{P}(A \cap B)$, $\mathbb{P}(A|B)$ and $\mathbb{P}(B|A)$ and verify these values by means of simulation.

3. (Coin experiment) Consider the probability space model of tossing a fair coin twice, i.e. a uniform probability measure on $\Omega = \{HH, HT, TH, TT\}$, where H indicates heads and T indicates tails. Simulate draws from this probability space and verify that the events "H appears on the first toss", "H appears on the second toss", and "both tosses have the same outcome" each have probability $1/2$.

(3) Random variables

1. Simulate the probability space model of throwing to dice and the random variable corresponding the sum of the pips. Visualize a normalized histograms of simulated outcomes of this random variable and compare it to the theoretical prediction.

2. Visualize the PMF of a Bernoulli random variable and a normalized histogram of many samples of a Bernoulli random variable with identical parameter setting on top of each other.

3. Visualize the PDF of a Gaussian random variable and a normalized histogram of many samples of a Gaussian random variable with identical parameter settings on top of each other.

(4) Joint distributions

1. Write a simulation that demonstrates that the marginal distributions of a bivariate Gaussian distribution with expectation parameter and covariance parameters

$$\mu = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ and } \Sigma = \begin{pmatrix} 0.3 & 0.2 \\ 0.2 & 0.5 \end{pmatrix}, \tag{1}$$

respectively, are given by univariate Gaussian distributions with expectation parameters $\mu_1 = 1, \mu_2 = 2$ and variance parameters $\sigma^2 = 0.3$ and $\sigma^2 = 0.5$, respectively.

2. Write a simulation that verifies that obtaining samples from 2 independent univariate Gaussian distributions with parameters $\mu_i, \sigma_i^2 > 0, i = 1, 2$ is equivalent to obtaining samples from a two-dimensional Gaussian distribution with the appropriately specified parameters $\mu \in \mathbb{R}^2$ and $\Sigma \in \mathbb{R}^{2\times 2}$.

3. Write a simulation that exemplary verifies the analytical results on conditional Gaussian distributions for the case of a bivariate Gaussian distribution.

(5) Transformations

1. Write a program that generates pseudo-random numbers from an exponential distribution using a uniform pseudo-random number generator and the probability integral transform theorem.

2. Let $X \sim N(0,1)$ and let $Y = \exp(X)$. Evaluate the PDF of Y analytically and verify your evaluation using a simulation based on drawing random numbers from $N(0,1)$.

3. Let $X \sim N(0,1)$ and let $Y = X^2$. By simulation, validate that Y is distributed according to a chi-squared distribution with one degree of freedom. Next, let $X_1, ..., X_{10} \sim N(0,1)$ and let $Y = \sum_{i=1}^{10} X_i^2$. By simulation, validate that Y is distributed according to a chi-squared distribution with ten degrees of freedom.

(6) Expectation and covariance

1. Sample $n = 10$ data points of a univariate Gaussian distribution and evaluate the sample mean, sample variance, and sample standard deviation.

2. Sample $n = 10$ data points of a bivariate Gaussian distribution and evaluate the sample covariation and sample correlation.

3. Validate the theorem on the variances of sums and differences of random variables using a sampling approach in a bivariate Gaussian scenario.

(7) Inequalities and limits

1. Write simulations that validate the Markov and Chebychev inequalities.

2. Write a simulation that validates the Weak Law of Large Numbers.

3. Write a simulation that validates the Lindenberg-Lévy Central Limit Theorem.

4. Write a simulation that validates the Liapunov Central Limit Theorem.

(8) Maximum likelihood estimation

1. Let $X_1, ..., X_n \sim$ Bern(μ) be $n = 20$ i.i.d. Bernoulli random variables. Using an optimization routine of your choice, formulate and implement the numerical maximum likelihood estimation of μ for true, but unknown values of $\mu = 0.7$ and $\mu = 1$ based on $X_1, ..., X_n$.

2. Let $X_1, ..., X_n \sim$ Bern(μ). For a large number n, sample the $X_1, ..., X_n$ and evaluate the maximum likelihood estimator $\hat{\mu}^{ML}$. Repeat this m times and create a histogram of the realized $\hat{\mu}_1^{ML}, ..., \hat{\mu}_m^{ML}$.

(9) Finite estimator properties

1. For $X_1, ..., X_n \sim$ Bern(μ) implement a simulation which validates the unbiasedness of the sample mean, the unbiasedness of the sample variance, the biasedness of the sample standard deviation, and the biasedness of the maximum likelihood variance parameter estimator.

2. For $X_1, ..., X_n \sim N(\mu, \sigma^2)$ implement a simulation which validates the unbiasedness of the sample mean, the unbiasedness of the sample variance, the biasedness of the sample standard deviation, and the biasedness of the maximum likelihood variance parameter estimator.

(10) Asymptotic estimator properties

1. Write a simulation that verifies the asymptotic unbiasedness of the maximum likelihood estimator for the variance parameter of a univariate Gaussian distribution. Include a verification of the unbiasedness of the sample variance.

2. Write a simulation that verifies the asymptotic efficiency of the maximum likelihood estimator for the parameter of a Bernoulli distribution.

3. Write a simulation that verifies the asymptotic efficiency of the maximum likelihood estimator for the variance parameter of a univariate Gaussian distribution.

(11) Confidence intervals

1. Write a simulation that verifies that the T statistic is distributed according to a t-distribution with $n - 1$ degrees of freedom.

2. Write a simulation that verifies that the 95%-confidence interval for the expectation parameter of a Gaussian distribution with unknown variance comprises the true, but unknown, expectation parameter in $\approx 95\%$ of its realizations.

3. Write a simulation that verifies that the approximate 95%-confidence interval for the expectation parameter of a Bernoulli distribution comprises the true, but unknown, expectation parameter in $\approx 95\%$ of its realizations.

(12) Hypothesis testing

1. By means of simulation, show that a two-sided T test with simple null hypothesis $\Theta_0 := \{\mu_0\}$ of significance level α' is exact.

2. By means of simulation, demonstrate that the δ-confidence interval-based test for the expectation parameter of univariate Gaussian distribution is of significance level $\alpha' = 1 - \delta$.

(13) Conjugate inference

1. For $n = 10$, implement batch and recursive Bayesian estimation for the Beta-Binomial model. Compare the results based on identical samples.

(14) Numerical methods

1. Estimate the expected value of a Beta(α, β) for varying values of α and β by means of Monte Carlo integration by using a Beta distribution random number generator. Compare the results to the true expected values.

2. Estimate the expected value of a Beta(α, β) for varying values of α and β by means of Monte Carlo integration using an importance sampling scheme and a uniform random number generator.

3. Use an acceptance-rejection algorithm to sample random numbers from Beta$(2, 6)$.